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Technical Note

The onset of thermal convection in an initially, stably stratified fluid layer

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1. Introduction

The system considered here is an initially quiescent, horizontal fluid layer of depth d, as shown in Fig. 1. The fluid layer is stratified stably with a uniform temperature gradient for time $t < 0$. The constant bottom temperature is kept at T_i with the upper constant surface temperature T_u ($>T_i$). With time $t \geq 0$ the layer is heated isothermally from below with the stepwise temperature increase to the constant temperature T_b. For a high ΔT ($=T_b-T_i$) buoyancy-driven convection will set in. The problem is to find the characteristic time to mark the onset of convective motion. Conventionally there are two characteristic times: one is the critical time to mark the onset of convective motion and the other is the time to represent manifest convection observable. Here the former one is denoted as t_c and the latter as t_0.

The important parameters to describe the present system are the Prandtl number Pr, the Rayleigh number Ra, and the temperature ratio γ defined as

$$Pr = \frac{\nu}{\alpha}, \quad Ra = \frac{g\beta \Delta T d^3}{\alpha\nu} \quad \text{and} \quad \gamma = \frac{T_u - T_i}{\Delta T} \tag{1}$$

where ν denotes the kinematic viscosity, α the thermal diffusivity, g the gravitational acceleration, and β the thermal expansion coefficient. In case of slow heating the basic temperature profile becomes linear, i.e. a constant temperature gradient and convection sets in when the Rayleigh number exceeds the critical value of

$$Ra(1-\gamma) = 1708 \tag{2}$$

But for a rapidly heated system of large Ra, the basic conduction state involves time and therefore, the related stability problem becomes very complicated.

The above situation is similar to the case of double-diffusive convection in initially stably stratified fluid layers [1,2]. Its related stability analysis was conducted by Ueda et al. [3]. Here their theoretical results from the amplification theory [4] and experimental ones will be compared with those from the frozen-time model [5] and also the propagation theory we have developed [1,6,7].

2. Stability analysis

For the present system we define a set of nondimensional variables τ and z by using the scale of time d^2/α and length d. The dimensionless basic temperature $\theta_0[=(T_0-T_u)/(T_b-T_i)]$ of the conduction state, can be described by

$$\theta_0 = (1-\gamma)(1-z) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\, n\pi z}{n}\exp(-n^2\pi^2\tau) \tag{3}$$

For deep-pool systems of small τ, the Levêque-type solution can be obtained as follows:

$$\theta_0 = 1 - \mathrm{erf}\left(\frac{z}{2\sqrt{\tau}}\right) - \gamma(1-z) \tag{4}$$

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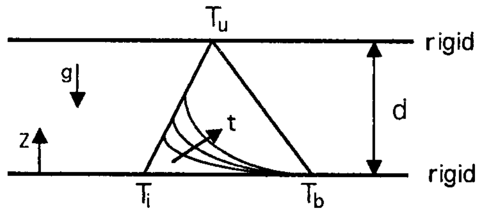

Fig. 1. Schematic diagram of the system considered here.

The above solution agrees well with Eq. (3) in the region of $\tau < 0.05$.

Under the linear stability theory the disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component w_1 under the Boussinesq approximation [1,2,7]:

$$\left(\frac{1}{Pr}\frac{\partial}{\partial\tau}-\nabla^2\right)\nabla^2 w_1=\nabla_1^2\theta_1 \tag{5}$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra\,w_1\frac{\partial\theta_0}{\partial z}=\nabla^2\theta_1 \tag{6}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\quad\text{and}\quad\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

The velocity disturbance w_1 has the scale of α/d and the temperature disturbance θ_1 has the scale of $\alpha\nu/(g\beta d^3)$. The proper boundary conditions are given by

$$w_1=\frac{\partial w_1}{\partial z}=\theta_1=0\quad\text{at}\ z=0\quad\text{and}\quad z=1 \tag{7}$$

The boundary conditions represent no flow on the boundaries, i.e. no-slip and the fixed temperature on the upper and lower boundaries. Our goal is to find the critical time τ_c to mark the onset of convection for a given Ra by using Eqs. (5)–(7). With the frozen-time model the terms involving $\partial(\cdot)/\partial\tau$ are neglected and the time becomes the parameter. With the amplification theory the proper initial conditions at $\tau=0$ are required and the amplification ratio to decide the time 't_0' to mark manifest convection becomes the most important parameter. The amplification theory is quite popular but its amplification ratio to represent manifest convection should be decided experimentally. However, the propagation theory described below is a rather simple, deterministic method even though it involves the time evolution.

According to the normal mode analysis, convective motion is assumed to exhibit the horizontal periodicity. Then the perturbed quantities can be expressed as follows:

$$\begin{aligned}&[w_1(\tau,\ x,\ y,\ z),\ \theta_1(\tau,\ x,\ y,\ z)]\\&\quad=[\overline{w}_1(\tau,\ z),\ \overline{\theta}_1(\tau,\ z)]\exp[\mathrm{i}(a_x x+a_y y)]\end{aligned} \tag{8}$$

where 'i' is the imaginary number and a_x and a_y represent the wave numbers. Substituting the above Eq. (8) into Eqs. (5)–(7) produces the usual amplitude functions in terms of the dimensionless horizontal wave number $a=(a_x^2+a_y^2)^{1/2}$. The propagation theory employed to find the critical time 't_c' to mark the onset of convective motion is based on the assumption that at the onset of convection, disturbances are propagated mainly within the dimensional thermal penetration depth Δ_T and Choi et al.'s [1,6] scale analysis would be valid for perturbed quantities. Then the following relationship is obtained:

$$\left|\frac{w_1}{\theta_1}\right|\sim\delta_T^2 \tag{9}$$

where δ_T $(=\Delta_T/d\propto\sqrt{\tau})$ is the dimensionless thermal penetration depth.

Now, the dimensionless amplitude functions of disturbances are expressed as

$$[\overline{w}_1(\tau,\ z),\ \overline{\theta}_1(\tau,\ z)]=[\tau w^*(\zeta),\ \theta^*(\zeta)] \tag{10}$$

where $\zeta=z/\sqrt{\tau}$. By using these relations the stability equations are obtained from Eqs. (5) and (6) as

$$\begin{aligned}&(D^2-a^{*2})^2w^*\\&\quad=a^{*2}\theta^*-\frac{1}{Pr}\left[\frac{1}{2}\zeta D^3w^*-\frac{1}{2}a^{*2}\zeta Dw^*+a^{*2}w^*\right]\end{aligned} \tag{11}$$

$$(D^2-a^{*2})\theta^*=-\frac{1}{2}\zeta D\theta^*+Ra^*\,w^*D\theta_0 \tag{12}$$

with the following boundary conditions,

$$w^*=Dw^*=\theta^*=0\quad\text{at}\ \zeta=0\quad\text{and}\quad 1/\sqrt{\tau} \tag{13}$$

where

$$a^*=a\sqrt{\tau},\quad Ra^*=\tau^{3/2}Ra,\quad D=d/d\zeta,\quad \gamma^*=\gamma\sqrt{\tau}$$

$$\text{and}\quad D\theta_0=\gamma^*-\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{4}\right)$$

For this deep-pool system of a given Pr and γ^*, a^* and Ra^* are obtained under the principle of the exchange of stabilities. In this case the outer boundary $\zeta=1/\sqrt{\tau}$ is practically equivalent to an infinite high value since τ is small. In other words the minimum

Table 1
Critical conditions for $\tau_c < 0.05$ from the propagation theory

Pr	γ^*	Ra_c^*	a_c^*	Pr	γ^*	Ra_c^*	a_c^*
0[a]	0.0	16.61	0.82	10	0.0	23.57	0.58
	0.05	20.41	0.89		0.05	43.25	0.76
	0.1	25.52	0.96		0.1	71.68	0.88
	0.15	32.64	1.05		0.15	117.08	0.99
	0.2	42.93	1.14		0.2	194.42	1.10
0.01	0.0	1799.1	0.82	100	0.0	20.70	0.54
	0.05	2228.6	0.89		0.05	41.35	0.76
	0.1	2812.2	0.98		0.1	69.62	0.88
	0.15	3632.4	1.07		0.15	114.65	0.99
	0.2	4833.5	1.17		0.2	191.14	1.10
0.1	0.0	219.10	0.81	1000	0.0	20.69	0.53
	0.05	273.83	0.89		0.05	41.16	0.76
	0.1	362.63	0.98		0.1	69.42	0.88
	0.15	484.69	1.06		0.15	114.41	0.99
	0.2	671.43	1.16		0.2	191.10	1.10
1	0.0	44.81	0.63	∞	0.0	20.67	0.53
	0.05	65.20	0.80		0.05	41.15	0.76
	0.1	96.40	0.90		0.1	69.40	0.88
	0.15	146.21	0.99		0.15	114.39	1.05
	0.2	230.16	1.10		0.2	191.07	1.10

[a] In the case of $Pr = 0$, Ra_c^* must be replaced by $Pr\ Ra_c^*$.

value τ_c and its corresponding wave number a_c are obtained for a given γ, Ra and Pr. For $Pr \to 0$, the left term in Eq. (11) is neglected due to insignificant viscous effects and therefore, $Pr\ Ra^*$ becomes the parameter. The above procedure is the essence of our propagation theory. Therefore, our propagation theory is a relaxed frozen-time model involving the terms $\partial(\cdot)/\partial\tau$ in Eqs. (5) and (6). The present analysis is to find the fastest growing disturbances at τ_c to a certain degree.

With an increase in time the basic temperature profile approaches a linear one, as is shown in Eq. (3). This corresponds to the case of very slow heating. With the linear basic temperature profile, many researchers have conducted the stability analysis. For a slightly nonlinear basic temperature field of large τ, the so-called frozen-time model is applied and therefore, it is assumed that $\partial w_1/\partial\tau = \partial\theta_1/\partial\tau = 0$, as usual. Now, the stability Eqs. (5) and (6) are independent of the Prandtl number and θ_0 is given by Eq. (3). The boundary conditions are given as usual. From these equations τ_c and a_c are found for a given γ and Ra. With this model $\overline{w}_1$ and $\overline{\theta}_1$ are functions of z only, for τ_c is the parameter independent of Pr.

The above equations were solved numerically by employing the outward shooting scheme [1,6].

3. Results and discussion

The critical values of Ra_c^* and a_c^* from the propagation theory are summarized in Table 1 for a given Pr and γ^*. It seems evident that Ra_c^* increases with a decrease in Pr and the Pr-effect on the critical condition is negligible for $Pr \geq 10$. The Pr-effect becomes pronounced for $Pr \leq 1$. The peculiar overall stability criteria for $Pr \to \infty$ are represented in Fig. 2. For $\tau_c < 0.04$ the critical time τ_c from the frozen-time model is lower than that from the propagation theory. Furthermore, the frozen-time model is independent of Pr and therefore, the difference between two models becomes larger with a decrease in Pr. Both models produce the minimum Ra and at $\tau_c \approx 0.1$ two curves do not intersect each other with $\gamma \geq 0.7$. In this γ-range the former predictions have been plotted up to its minimum Ra-value and in the domain exceeding its corresponding τ_c the latter ones have been plotted. The propagation theory would be valid for small τ_c while for large τ_c the frozen-time model is valid. Ra approaches the infinite value at

$$\tau_c \cong \gamma^2/\pi \quad \text{for } \gamma > 1 \tag{14}$$

This condition is obtained from Eq. (4) with the stable condition $\partial\theta_0/\partial z|_{z=0} > 0$, which means that the hori-

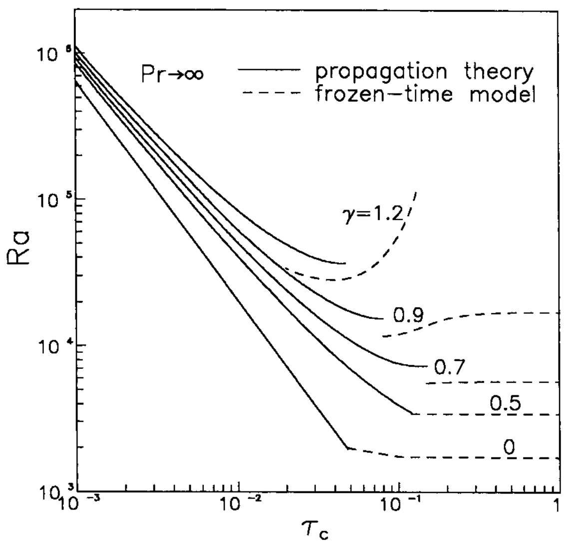

Fig. 2. Predictions of the critical time to mark the onset of convective motion.

zontal fluid layer is absolutely stable due to cooling from below. The similar trend can also be seen from the propagation theory. For a given Pr and Ra the critical values of τ_c increase with increasing γ. All these results will be valid for $\tau_c < 0.05$. The amplitude profiles calculated show that the temperature disturbances are confined mainly to some appreciable thermal depth while the velocity disturbances are propagated more deeply from the heated surface with an increase in Pr. It is certain that the inertial terms make the system more stable and for $Pr < 1$ both the temperature and velocity disturbances are propagated within almost the same thermal thickness. This trend characterizes the boundary layer flow of buoyancy-driven convection.

For the basic state of nearly linear temperature profiles, i.e. sufficiently large τ_c, the frozen-time model can be a good approximation. In case of $\tau \to \infty$, the stability limits represented by Eq. (2) are independent of Pr. For $\gamma \geq 1$ the instabilities will disappear with increasing time, e.g. $\tau_c > 0.1$. It is interesting that some kind of subcritical state is possible for $\gamma > 0.7$, as shown in Fig. 2. The larger the value of γ, the more easily the minimum value of Ra near $\tau_c = 0.05$ is detected. With $\gamma = 0.9$ amplitude profiles show double cell characteristics at $\tau_c = 0.01$. This peculiar behavior seems to reflect the possibility of multiple cell patterns. These interesting phenomena are also discussed in Ueda et al.'s [3] work. With $\gamma = 1.2$ the system illustrated in Fig. 2 will experience two critical times for a given Ra. For example, in the case of $Ra = 10^5$, the first instabilities will appear at $\tau_{c_1} \cong 0.01$ with the second ones at $\tau_{c_2} \cong 0.1$. τ_{c_2} may be meaningless because of the incipient convective motion at τ_{c_1}. Of course, these

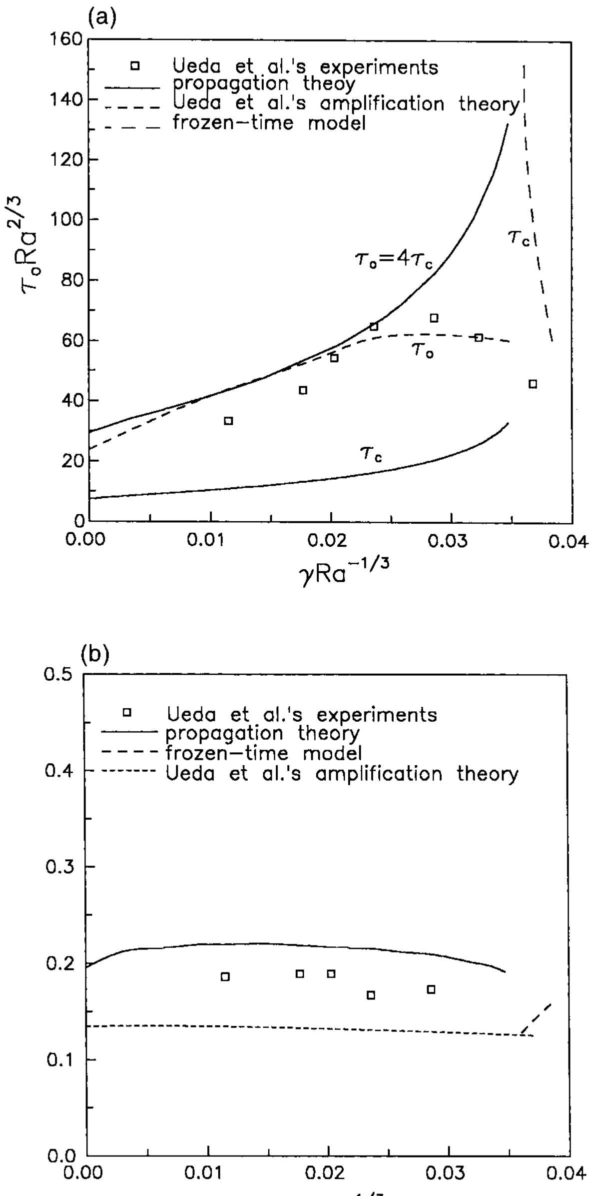

Fig. 3. Comparison of predictions (amplification theory with Pr = 8800 and $\hat{E} = 10^2$; propagation theory with $Pr \to \infty$) at Ra = 15,000 with experimental ones of Ra = 9000–17,000 and Pr = 8800: (a) characteristic times; and (b) associated critical wave numbers. Here τ_c and τ_0 denote the characteristic time to mark the onset of convection and that of manifest convection, respectively.

instabilities will disappear in the range exceeding the time given by Eq. (14). Near $\tau_c = 0.1$ the predictions from two models are not exact. But for $\gamma \leq 0.5$ the two predictions intersect each other and it is interesting that for $\tau_c > 0.1$ Ra is given by Eq. (2). Therefore, it may be loosely stated that the predictions in Fig. 2 cover the whole domain for $Pr \rightarrow \infty$.

With decreasing Pr the τ_c-value from the propagation theory will increase. Choi et al. [7] presented an approximate correlation of

$$\tau_c = 7.80[(1 + 0.739/Pr)/Ra]^{2/3} \quad \text{for} \quad \gamma = 0 \tag{15}$$

which agrees well with those in Table 1. Lee [8] and Patrick and Wragg [9] conducted electroplating experiments of mass transfer using the limiting current method. The deviation points of mass transfer from that of mass diffusion agree well with those values of $4\tau_c$ obtained from Eq. (15) with $Pr \rightarrow \infty$. Foster [4] commented that a certain growth time of disturbances is required to be detected experimentally and the relation of $t_0 \cong 4t_c$ would be maintained. This means that manifest convection will be exhibited at $t = t_0 > t_c$ in the deep-pool systems of $\gamma = 0$ [6,10,11].

For analyzing instabilities of the present system Ueda et al. [3] conducted experiments and the experimental data points were compared with predictions from the amplification theory. They expanded the temperature and vertical velocity disturbances into a series of orthonormal functions which satisfy the boundary conditions automatically. For example, the temperature disturbance was represented by

$$\overline{\theta}_1 = \sum_{1}^{\infty} A_n(\tau) \sin\, n\pi z \tag{16}$$

with $A_n(0) = 1$ and the amplification ratio $\hat{E}$ was defined as

$$\hat{E} = \frac{[\text{total energy at } \tau]}{[\text{total energy at } \tau = 0]} \tag{17}$$

For details, refer to their work. With these the fastest growing disturbance and its characteristic time to reach the predetermined $\hat{E}$-value, i.e. 1, 10^2, 10^4 or 10^8 was found. It is doubtful whether the temperature disturbances like Eq. (16) will exist at $t = 0$. But it will be interesting to compare their predictions with the present ones. In the case of $Pr = 7$, the present τ_c is lower than theirs and the present τ_0 $(= 4\tau_c)$ locates between $\hat{E} = 1$–10^2. It is noted that their amplification theory is rather sensitive to Pr but with the propagation theory the effect of Pr is negligible for $Pr \geq 10$.

In experiments Ueda et al. [3] used very viscous fluid of $Pr = 8800$. In their experiments Ra ranged from 9000–17,000 with $\gamma = 0.73$–1.67. For comparison with experiments they employed the amplification theory with $Ra = 15{,}000$ and $Pr = 150$, while the present predictions obtained from the propagation theory are those of $Ra = 15{,}000$ and $Pr \rightarrow \infty$. Fig. 3(a) shows that the present τ_c-values are about one-fourth of their theoretical predictions of $\hat{E} = 10^2$ for $\gamma\ Ra^{-1/3} < 0.03$ and in this range theirs agree well with those from $\tau_0 = 4\tau_c$. Since the Rayleigh number is fixed at 15,000 in the theoretical analyses, predictions of τ_0 and a_c in Fig. 3 reflect the γ-effect only. τ_0 shows the maximum near $\gamma\ Ra^{-1/3} = 0.028$ with experiments and if it is converted with $Ra = 15{,}000$, near $\gamma = 0.7$ from the amplification theory. But the propagation theory shows that the values of $\tau_0\ Ra^{2/3}$ increase with increasing $\gamma\ Ra^{-1/3}$ and terminates at $\tau_c \cong 0.1$, as shown in Fig. 2. With $Pr = \infty$ and $\gamma = 0$, the relation of $\tau_0\ Ra^{2/3} \cong 30$ is obtained from Eq. (15) or Table 1. The last two experimental data points for $\gamma\ Ra^{-1/3} > 0.03$ deviate significantly from results obtained from the propagation theory. In Fig. 3(a) the τ_{c_2}-values of the second instabilities from the frozen-time model (see Fig. 2) are also plotted in the range of $0.035 < \gamma\ Ra^{-1/3} < 0.0395$. Based on Eq. (2), it is known that the value of $\gamma\ Ra^{-1/3}$ terminates at 0.0395 since the value of $\gamma\ Ra^{-1/3}$ reaches the maximum with $\gamma = 0.75$. In other words the system having a linear temperature gradient in the fully-developed conduction state is stable for $\gamma\ Ra^{-1/3} > 0.0395$. This means that the growth period from the onset of convective motion to detection of manifest convection decreases in the range exceeding a certain γ-value and the relation of $\tau_c < \tau_0 < 4\tau_c$ is possible. Therefore, the last experimental data point in Fig. 3(a) may be close to the predicted τ_c-values from both the frozen-time model and the propagation theory. But it may be stated that τ_c from the propagation theory produces a minimum bound of detection time τ_0.

It is interesting that with the frozen-time model τ_c decreases with increasing γ. But predictions are still far from experimental data. The difference may be caused by the following. The above predictions of τ_c for $Ra = 15{,}000$ are those for an initially linear temperature profile but Ueda et al.'s experimental one was S-shaped with the lower half slightly warmer than the linear one and the upper half cooler. Furthermore, the definition of the critical time is not clear for the linear temperature profile. In other words the critical time may be meaningless for large time, wherein τ_c varies widely with a small change in Ra, as shown in Fig. 2. For small time Yoon et al. [2] reported that strong stratification, i.e. a high γ-value brings early manifest convection of $\tau_0 \cong \tau_c$. Another comparison is possible with respect to the critical wave number, as shown in Fig. 3(b). The cell size obtained from the propagation theory is smaller than that from the amplification theory and experimental data points locate between predictions from two models. All the above comparison concludes that Ueda et al.'s [3] experimental range is in the transition region from the deep-pool to the large-τ_c system owing to rather low Ra-values. It seems certain that the present propagation theory is a good method to predict the onset of thermal convection in a horizontal fluid layer of large *Ra*, i.e. in the deep-pool system of small time.

4. Conclusion

The onset of buoyancy-driven motion in initially stably-stratified horizontal fluid layers has been analysed here by using the linear stability theory. For small time, the propagation theory has been employed to predict the critical time to mark the onset of convective motion and for large time the frozen-time model has been used. It is shown that for $\gamma\ Ra^{-1/3} < 0.03$ the propagation theory produces the stability criteria consistent with Ueda et al.'s [3] experiments and predictions from the amplification theory. But the frozen-time model produces a peculiar shape in the plot of $\tau_0\ Ra^{2/3}$ vs $\gamma\ Ra^{-1/3}$. For large γ-values both the multiple-cell patterns and the behavior of subcritical state are exhibited, producing the minimum of *Ra* in the plot of *Ra* vs τ_c for a given γ. In this γ-range two critical times exist for a given *Ra* and *Pr*. The earlier one brings convective motion and for $\gamma \geq 1$ motion will disappear with time. But the later one is possible in case of very weak motion. Finally, no motion will exist in this γ-range. The incipient cell pattern is almost independent of γ and for large time the critical time loses its role as a parameter.

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